APPLICATION OF THE MULTIGRID METHOD IN NUMERICAL SIMULATION OF SEDIMENT TRANSPORT

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ABSTRACT

By combining FAS (Full Approximate Scheme) for nonlinear equations of multigrid method with Finite Volume Method, a plane two-dimensional mathematical model of sediment-transport is presented in this paper. The analysis on the efficiency of this model shows that it can reduce the amount of computation greatly. Numerical simulation results of the Yellow River Langdian Water Source Project agree well with the test results of the physical model.

Key Words: Multigrid method, Mathematical model, Sediment transport

1 INTRODUCTION

The multigrid method (Brandt, 1988) is widely used in many disciplines and engineering subjects, especially in the area of Computational Fluid Mechanics because of its high efficiency in computation. The multigrid method has been proven in theory to be one of best numerical methods for solving linear elliptic problems because the load of computation is in direct proportion to the number of grid nodes; the convergent speed of multigrid method is irrelevant to the size of grid. So it is suitable for the numerical simulation of full scale engineering projects. The multigrid method can accelerate the computation program by 10 to 100 times, (Liu, 1995) thus the multigrid method presents a solution to the problem of the enormous computation load and the relative low speed in the simulation of sediment transport.

Since the late 1980’s achievements in multigrid method are: the rigorous justification of the local mode analysis, including the proof that the exact operation count per grid point was predictable and essentially independent of boundary shapes and boundary conditions; the treatment of non-elliptic steady state problems; and extremely efficient multilevel techniques for time-dependent problems etc. (Brandt, 1988)

As the theory of the multigrid method was matured, the applications multigrid method have been gradually extended to the computation of fluid dynamics, structural mechanics and electromagnetism, etc. The multigrid method was mostly combined with FDMs or FEMs and is now combined with FVMs for their incomparable advantages. Recently, Multilevel Adaptive Grids and Parallel Algorithms were introduced into numerical simulation techniques, which improved the efficiency and application of the multigrid method greatly (Liu, 1995).

As mentioned above, the multigrid method is widely used in many disciplines for its high efficiency in computation. However, its application in the numerical simulation of sediment transport has not been seen before. This paper will establish the plane two-dimensional mathematical model of flow and sediment transport using the multigrid method and then explore its advantages by analyzing computational results in comparison to a similar model to lay a foundation for its further application in the numerical simulation of sediment transport.

The Langdian Water Source Project will be built in the middle reach of the Yellow River. Because of the high concentration of sediment, the channel is often filled up by sediment. The purpose of the computation is to simulate the scour on shoal between the main stream and the station to find out whether the scour channel can cross the shoal to get sufficient discharge for the pumping station.

2 GOVERNING EQUATIONS

The governing equations for the plane two-dimensional flow are as follows: (Xie, 1990)

The continuity equation:

\[ \frac{\partial h}{\partial t} + \frac{\partial (hu)}{\partial x} + \frac{\partial (hv)}{\partial y} = 0 \]  

(1)

The momentum equations:

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\[
\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} + g \frac{\partial z}{\partial x} + g \frac{n^2 u v^2}{h^{4/3}} = \nu \left( \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right)
\]

(2)

\[
\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} + g \frac{\partial z}{\partial y} + g \frac{n^2 v u^2}{h^{4/3}} = \nu \left( \frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} \right)
\]

(3)

where \( u \) and \( v \) = velocity components of flow; \( n \) = roughness of riverbed; \( h \) = water depth; \( z \) = water level; \( g \) = acceleration of gravity and \( \nu \) = dynamic viscosity coefficient.

3 NUMERICAL SIMULATION PROCEDURE

Two-dimensional open channel shallow water equations are quite similar to two-dimensional incompressible N-S equations, so the numerical methods used in solving N-S equations can be transplanted to solve the shallow water equations. This paper will employ the numerical methods presented by Liu (1995) for the N-S equations to discrete and solve the shallow water equations.

3.1 Equation Discretization

The FVM is employed in this paper to discrete the governing equation because of its fitness for the application of multigrid method besides such advantages as it can automatically satisfy mass conversation and easily establish boundary conditions, etc. And the staggered grid technique, which the velocity components are saved at the boundary of grid and the pressure items are saved at the center of grid, is also used to avoid the decoupling between the grids which would cause the unstability of the scheme. Thus the discrete equations as follows are obtained:

The discrete continuity equation:

\[
C_{wE} u_c + C_{wE} u_E + C_{wN} v_c + C_{wN} v_N + F_c h_c = S_w
\]

(4)

where

\[
C_{wE} = \frac{h_c^0 - h_w^0}{2\Delta x}
\]

\[
C_{wE} = \frac{h_c^0 + h_w^0}{2\Delta x}
\]

\[
C_{wN} = \frac{h_c^0 + h_w^0}{2\Delta x}
\]

\[
C_{wN} = \frac{h_c^0 - h_w^0}{2\Delta x}
\]

\[
F_c = \frac{1}{\Delta t}
\]

\[
S_w = \frac{1}{\Delta t} h_c^0
\]

where the superscript "0" represents the value of last time step and the subscripts C, N, S, E, S represent the values of the central, the north, the south, the east and the west point respectively.

The momentum equations:

\[
u \text{ component:}

A_R u_E + A_W u_W + A_N u_N + A_S u_S - A_c u_c + D_w Z_W - D_c Z_C = S_u
\]

(5)

\[
v \text{ component:}

B_R v_E + B_W v_W + B_N v_N + B_S v_S - B_c v_c + E_s Z_S - E_v Z_v = S_v
\]

(6)

where

\[
A_R = \max \left\{ \frac{v}{\Delta x^2} \frac{1}{2\Delta x} \left| u^{n+1}_c \right| \right\} - \frac{1}{2\Delta x} u^{n+1}_c
\]

\[
A_W = \max \left\{ \frac{v}{\Delta x^2} \frac{1}{2\Delta x} \left| u^{n+1}_w \right| \right\} + \frac{1}{2\Delta x} u^{n+1}_w
\]
\[ A_N = \max \left( \frac{v_{\perp}}{\Delta y^2}, \frac{1}{2\Delta y} \right) \frac{v_{\perp}^*}{\Delta t} - \frac{1}{2\Delta y} v_{\perp}^* \]
\[ A_s = \max \left( \frac{v_{\perp}}{\Delta y^2}, \frac{1}{2\Delta y} \right) \frac{v_{\perp}^*}{\Delta t} - \frac{1}{2\Delta y} v_{\perp}^* \]
\[ A_c = A_e + A_w + A_N + A_s + \frac{1}{\Delta t} \]
\[ D_w = D_c = g \frac{1}{\Delta x} \]
\[ S_w = gn^2 \left( \frac{u_c \sqrt{u_c^2 + v_c^2}}{h_c^{1/3}} \right) - \frac{u_c}{\Delta t} \]
\[ B_e = \max \left( \frac{v_{\perp}}{\Delta x^2}, \frac{1}{2\Delta x} \right) \frac{v_{\perp}^*}{\Delta t} - \frac{1}{2\Delta x} v_{\perp}^* \]
\[ B_w = \max \left( \frac{v_{\perp}}{\Delta x^2}, \frac{1}{2\Delta x} \right) \frac{v_{\perp}^*}{\Delta t} + \frac{1}{2\Delta x} v_{\perp}^* \]
\[ B_N = \max \left( \frac{v_{\perp}}{\Delta y^2}, \frac{1}{2\Delta y} \right) \frac{v_{\perp}^*}{\Delta t} - \frac{1}{2\Delta y} v_{\perp}^* \]
\[ B_s = \max \left( \frac{v_{\perp}}{\Delta y^2}, \frac{1}{2\Delta y} \right) \frac{v_{\perp}^*}{\Delta t} - \frac{1}{2\Delta y} v_{\perp}^* \]
\[ B_c = B_e + B_w + B_N + B_s + \frac{1}{\Delta t} \]
\[ E_s = E_c = g \frac{1}{\Delta y} \]
\[ S_c = gn^2 \left( \frac{v_c \sqrt{u_c^2 + v_c^2}}{h_c^{1/3}} \right) - \frac{v_c}{\Delta t} \]

### 3.2 Solution Algorithm

Generally, there are two methods to solve the incompressible N-S equations: one is the Chorin Artificial Compression method, i.e. to solve incompressible problems as compressible problems; the other is the Spalding Modified Pressure method, i.e. SIMPLE method. But these two methods don't fit the multigrid method well. However, the Approximate Line-Box Relaxation method (ALB) fit the multigrid method better than the two methods mentioned above. (Liu, 1995) According to above stated, the shallow-water equations are similar to the 2D incompressible N-S equations. So this paper is to employ the ALB method to solve the discrete equations. The ALB method need to solve all boxes along a grid line (x-direction or y-direction) at the same time (see also Fig.1). Suppose there are unknown quantities \( u_i^j, \ v_i^j, \ v_i^j, \ z_i^j \) \((j=2,3,\ldots,Ny-1)\) and then the corresponding discrete equations are obtained:

\[ A_E u_i^j + A_w u_i^j + A_N u_i^j + A_s u_i^j - A_e u_i^j + D_e Z_i^{j+1} - D_e Z_i^j = S_{uj} \]  
(7)
\[ B_E v_i^j + B_w v_i^j + B_N v_i^j + B_s v_i^j - B_e v_i^j + E_e Z_i^{j+1} - E_e Z_i^j = S_{vj} \]  
(8)
\[ C_w u_i^j + C_w u_i^j + C_s v_i^j + C_{ew} v_i^j + F_i Z_i^{j+1} = S_{uj} \]  
(9)

where the superscript represents the position of discrete point.

The detailed solution procedures are: fix \( z \) values and perform Gauss-Seidel interation for several times to satisfy all the momentum equations approximately. And then adjust \( u, \ v \) and \( z \) in every box to satisfy the continuity and momentum equations. Since the old values of \( u, \ v \) and \( z \) have satisfied the momentum equations approximately, they can be eliminated from the new equations. And then the updated equations are obtained:
\[
(A_{E}^j + A_{C}^j) \varepsilon_j - D_{E}^j \Delta Z_j = 0
\]
(10)
\[
(B_{E}^j + B_{C}^j) \delta_j - E_{E}^j \Delta Z_j = 0
\]
(11)
\[
\frac{2h_{C}^{old}}{\Delta x} e_j + \frac{h_{C}^{old}}{\Delta y} (-\delta_{j+1} + 2\delta_j - \delta_{j-1}) + \frac{\Delta h_j}{\Delta t} S_{mj} = 0
\]
(12)
where
\[
S_{mj} = - \left( \frac{u_{E}^{old} - u_{C}^{old}}{\Delta x} + \frac{v_{E}^{old} - v_{C}^{old}}{\Delta y} \right) h_{C}^{old} - \frac{h_{C}^{old}}{\Delta t}
\]
where \( j \) is the box serial number, \( j=2, 3, \ldots, N_{x,1} \). If the bed deformation is not taken into consideration, the following relation is obtained
\[
\frac{\Delta h_j}{\Delta t} = \frac{\Delta z_j}{\Delta t}.
\]
If the both sides of Equation (12) are divided by \( h_{C}^{old} \), we have
\[
\frac{2}{\Delta x} e_j + \frac{1}{\Delta y} (-\delta_{j+1} + 2\delta_j - \delta_{j-1}) + \frac{\Delta z_j}{h_{C}^{old} \Delta t} = \frac{S_{mj}}{h_{C}^{old}}
\]
(13)
From Equation (10) and (11), we have:
\[
\frac{\varepsilon_j}{\delta_j} = \frac{\Delta y}{\Delta x} \frac{B_{E}^j + B_{C}^j}{A_{E}^j + A_{C}^j} = \alpha
\]
and
\[
\frac{\Delta Z_j}{E_{C}^j} = \frac{B_{E}^j + B_{C}^j}{E_{C}^j} \delta_j = \beta \delta_j
\]
Substitute them to Equation (13), then
\[
\frac{2 \alpha}{\Delta x} \delta_j + \frac{1}{\Delta y} (-\delta_{j+1} + 2\delta_j - \delta_{j-1}) + \frac{\beta}{h_{C}^{old} \Delta t} \delta_j = \frac{S_{mj}}{h_{C}^{old}}
\]
(14)
If \( a_j = \left[ \frac{2 \alpha}{\Delta x} + \frac{2}{\Delta y} + \frac{\beta}{\Delta t \cdot h_{C}^{old}} \right] \)
\[
b_j = -\frac{1}{\Delta y}
\]
\[
c_j = -\frac{1}{\Delta y}
\]
then Equation (14) can be written into a tridiagonal matrix equation:
\[
\begin{bmatrix}
a_2 & b_2 & & & \\
c_3 & a_3 & b_3 & & \\
& \ddots & \ddots & \ddots & \\
& & c_{n-2} & a_{n-2} & b_{n-2} \\
& & & c_{n-1} & a_{n-1}
\end{bmatrix}
\begin{bmatrix}
\delta_2 \\
\delta_3 \\
\vdots \\
\delta_{n-2} \\
\delta_{n-1}
\end{bmatrix}
= \begin{bmatrix}
S_{m2}^* \\
S_{m3}^* \\
\vdots \\
S_{mn-2}^* \\
S_{mn-1}^*
\end{bmatrix}
\]
(15)
Since the matrix is a tridiagonal matrix, the Thomas algorithm can solve it. \( \varepsilon_j \) and \( \Delta z_j \) can be derived by
\[
\varepsilon_j = \alpha \delta_j
\]
and
\[
\Delta z_j = \beta \delta_j
\]
(16)
after $\delta_j$ is obtained. And then update $u$, $v$ and $z$ to satisfy all the momentum and continuity equations:

$$u^l_j \leftarrow u^l_j + \varepsilon_j$$
$$u^l_i \leftarrow u^l_i - \varepsilon_j$$
$$v^l_j \leftarrow v^l_j + \delta_j - \delta_{j+1}$$
$$v^l_i \leftarrow v^l_i - \delta_j + \delta_{j-1}$$
$$z^l_c \leftarrow z^l_c + \Delta z_j$$ (17)

4 APPLICATION OF MULTIGRID METHOD

Only uniform staggered grid is discussed in this paper. But non-uniform grids are employed in the program for its versatility. A sketch map of two-layer grids is shown in Fig.2.

Considering the nonlinearity of equations, the Full Approximate Scheme (FAS) is used in this paper. The FAS of a two-layer grid can be described concisely as follows:

1. Perform the ALB to the discrete equations on the fine grids for 1~2 times: $L_hW^h = f^h$
2. Solve the equation below on the coarser grids:
   $$L_{2h}W^{2h} = L_{2h}I_{2h}^{2h}W^h + \tilde{I}_{2h}^{2h}(f^h - L_hW^h)$$
3. Make correction on the coarser grids:
   $$W^h = W^h + I_{2h}^h(W^{2h} - I_{2h}^{2h}W^h)$$

where $I_{2h}^h$ is the limiting factor of variables, $\tilde{I}_{2h}^{2h}$ is the limiting factor of residuals, and $I_{2h}^{2h}$ is the bilinear interpolation factor. $R_u$ is the residual of momentum equation in x-direction, $R_v$ is the residual of momentum equation in y-direction and $R_m$ is the residual of continuity equation.

Fig.3 describes the limiting method for x-momentum equation and Fig.4 describes the interpolation method for $u$, $v$ and $z$.

The coarser grid correction includes the correction to $u$, $v$ and $z$:

$$u^h \leftarrow u^h + I_{2h}^h(u)[u^{2h} - I_{2h}^{2h}(u)u^h]$$
$$v^h \leftarrow v^h + I_{2h}^h(v)[v^{2h} - I_{2h}^{2h}(v)v^h]$$
$$Z^h \leftarrow Z^h + I_{2h}^h(Z)[Z^{2h} - I_{2h}^{2h}(Z)Z^h]$$ (18)

5 THE EFFICIENCY OF THE MULTIGRID METHOD

The multigrid method is efficient in computation, as shown in the following case study of the Yellow River Langdian Water Source Project. The computation is carried out under the negligible concentration and fixed bed. Fig.5 shows that the iteration times to satisfy the required error limit decreases when the layer of grid increases. If the layer is 5, the required interation time is only about one-sixth compared with that of one layer. So the multigrid method can improve the convergence of computation greatly.

If the required workload to execute a relaxation interation on the finest grids is a work unit (WU), for V(1,1) iteration, i.e. $V_1 = V_2 = 1$, the required workload on the finest grids would be $2$ WU. If $d$ is the dimensionality, the workload on the second layer would be $2 \times 2^{-d}$. On the analogy of that, the total workload would be

$$2(2^{-d} + 2^{-2d} + \cdots + 2^{-nd})WU < \frac{2}{1 - 2^{-d}}WU$$ (19)

where $n$ is the number of grid layers.

Formula (20) shows that the required workload of one-dimensional V(1,1) iteration is 4WU, that of two-dimensional problems is $\frac{8}{3}$ WU and it would be $\frac{16}{7}$ WU if $d = 3$. For the above example, the number of required interation times to satisfy the error is 127 if $n = 1$, i.e. the required workload is

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If \( n=5 \), the number of required iteration times is 17 and the required amount would be 
\[
\frac{8}{3} \times 17 = 45 \frac{1}{3}
\]
WU. Moreover, if the amount of interpolation and restriction between grids is taken into
consideration, which is about 15\%–20\% of the total amount, the total required amount would be no more
than 60WU, which is less than half of the single grid workload.

The analysis above shows that the multigrid method can improve the convergence speed, decrease the
number of iteration times and save the amount of computation greatly.

6 NUMERICAL SIMULATION OF SEDIMENT TRANSPORT

6.1 Governing Equations

The continuity equation of the suspended load:

\[
\frac{\partial S}{\partial t} + u \frac{\partial S}{\partial x} + v \frac{\partial S}{\partial y} = - \frac{\rho'}{h} \frac{\partial z_0}{\partial t} + e \left( \frac{\partial^2 S}{\partial x^2} + \frac{\partial^2 S}{\partial y^2} \right)
\]

(20)

The bed deformation equation:

\[
\alpha \omega (S - S_*) = \rho' \frac{\partial z_0}{\partial t}
\]

(21)

The flow capability:

\[
S_* = f(U, h, \omega, \ldots)
\]

(22)

where \( u \) and \( v \) are the velocity in the \( x \) and \( y \) direction respectively, \( S \) is the suspended load
centration, \( S_* \) is the capability of the flow, \( h \) is the water depth, \( \rho' \) is the sediment specific
density, \( z_0 \) is the elevation of bed surface, \( \omega \) is the particle settling velocity, \( \alpha \) is the saturation
recovery coefficient, \( U \) is the resultant velocity and \( \epsilon_s \) is the particle distribution coefficient.

The usually used formula for the uniform suspended load is given below:

\[
S_* = k \left( \frac{U^3}{gh\omega} \right)^m
\]

(23)

where \( k \) and \( m \) are the empirical coefficient and the empirical index number respectively.

6.2 The Discretization of Governing Equations

The backward difference method is employed in the paper to discrete the sediment continuity equation
and the bed deformation equation.

\[
S_c^{n+1} = \left( \frac{1}{\Delta t} S_c^n + \frac{1}{\Delta x} u_c S_w^{n+1} + \frac{1}{\Delta y} v_c S_s^{n+1} + \frac{1}{h_c} \alpha \omega S_{c^{n+1}} \right)
\]

(24)

\[
\Delta z_0 = \alpha \omega \Delta t (S_c^{n+1} - S_*^{n+1}) / \rho'
\]

(25)

6.3 Calculation Procedure

For the time step \( n+1 \),

1. Give the guessed values for \( u, v \) and \( z \).
2. Calculate \( u, v \) and \( z \) according to the procedure described in section 3.2.
3. Calculate \( S_*, S_c \) and the scour-and-deposit depth \( \Delta z \) according to Equation (23)-(25).
4. Correct the bed elevation \( z_h = z_h + \Delta z \).
5. Go to 1 for the next time step.
7 APPLICATION

The model will be applied to examine the flow field and potential bed deformation induced by the Yellow River Langdian Water Source Project under three sediment concentrations. The simulated results will be verified by the physical model test results. The Langdian Project will be built as the source for irrigation water in the region of the small north trunk stream of the Yellow River that will lie in the middle west of Yuncheng District, Shanxi province. The designed intake discharge is 42.9m³/s and the irrigated area is about 144.46 thousand hectares.

7.1 Simulation Conditions

The terrain and the boundary conditions used in the mathematical model refer to those used in the physical model test for the comparability between the numerical simulation results and the physical model results. The purpose of the computation is to simulate the scour on shoal between the main stream and the station at three different sediment concentration of the main flow: 10kg/m³, 30kg/m³ and 50kg/m³ and to find out whether the scour channel can cross the shoal to get sufficient discharge for the pumping station. The size of the computed region is the same as that of the physical model, i.e. 440×300m. The designed altitude of the shoal is 347.0m and the average altitude of the main channel bed is 346.13m. The size of the settling pool in front of the pumping station is 40×60m and the altitude of the bottom is 343.5m (see also Fig.6). The main flow discharge is Q=253m³/s when the level of main flow is 347.5m and the corresponding pumping discharge of the station is 42.9m³/s.

Because the shape of the simulated region is regular, the square grid is used in computation and the size of the finest grid is 10×10m, the total number of finest grids used is 44×30. The number of layers in computation is five. In order to show the figure clearly, only two layers of grid is given in the Fig.7. Boundary conditions encountered in the computation are: the inlet, the outlet, the water boundary and rigid wall. At the inlet, usually the known boundary values are prescribed. At the outlet and the water boundary, the gradients of the both velocity components and the water level are set to zero. At the rigid wall, both the two velocity components are set to zero. The calculated results are shown in Fig.8-12. The calculated results are in good agreement with the physical model test results.

8 CONCLUSION

1 The multigrid method is widely used in many engineering subjects because of its high efficiency in computation, but its application in the numerical simulation of sediment transport has not been seen before. The paper first narrated the advantages of the multigrid method and its range of applications, and then combined it with the FVM to build a plane two-dimensional mathematical model of sediment transport. Then employed the model to simulate the scour on the shoal near the pumping station of the Yellow River Langdian Water Source Project.

2 The high efficiency of the multigrid method was verified by employing the model to simulate the fixed bed clear water flow. The calculated results show that the multigrid method can decrease the amount of computation greatly comparing with the single grid method. If the layer number is set to 5, the computation amount of this example will less than half amount of the single grid.

3 The model was employed to simulate the scour on the shoal near the pumping station of the Langdian Project and the simulation was carried on three different conditions: the sediment concentration of the main flow is 10kg/m³, 30kg/m³ and 50kg/m³ respectively. The calculated results show that: if the sediment concentration of the main flow is 10kg/m³ or 30kg/m³, the cross channel through the shoal may be scoured to supply sufficient water for the pumping station. While if the sediment concentration is 50kg/m³, the shoal is choked heavily with sediment before the cross channel has come into being and the required discharge of the pumping station can not be fulfilled. The above computed results are in good agreement with that of the physical model test.

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![Fig.1 The approximate line-box relaxation method](image1)

![Fig.2 A two-layer uniform staggered grid.](image2)

![Fig.3 The limitation to x-momentum equation residuals](image3)

![Fig.4 The bilinear interpolation method](image4)
Fig. 5 The convergence speed comparison between multigrid and single grid

Fig. 6 The layout of the physical model test

Fig. 7 Grid used in the simulation
Fig. 8  The computed velocity field

Fig. 9  The contour map of the shoal scour in 24h (10kg/m²)

Fig. 10  The contour map of the shoal scour in 24h (30kg/m²)
Fig. 11  The contour map of the shoal scour in 24h (50kg/m³)

Fig. 12  The contour map of the physical model test result (50kg/m³)