THE STUDY ON THREE-DIMENSIONAL MATHEMATICAL MODEL OF RIVER BED EROSION FOR WATER-SEDIMENT TWO-PHASE FLOW

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ABSTRACT: Based on the tensor analysis of water-sediment two-phase flow, the basic model equations for clear water flow and sediment-laden flow are deduced in the general curve coordinates for natural water variable-density turbulent flow. Furthermore, corresponding boundary conditions are also presented in connection with the composition and movement of non-uniform bed material. The theoretical results are applied to the calculation of the float open caisson in the construction period and good results are obtained.

KEY WORDS: flow mathematical model, sediment-laden flow mathematical model, water-sediment two-phase flow, float open caisson

1 INTRODUCTION

The building of bridge-base in the river channel always breaks the relevant balance of flow, sediment transportation and river bed, the water-sediment movement and river bed form will also be readjusted. Generally, to build the base of bridge in deep water the construction method of float open caisson besides pile foundation is adopted. In the constructive period, parts of the float open caissons are precasted in the factory or workshop and are put together into boat, then they settle down slowly on the river bed. In the process of settlement, the flow, the sediment transportation and river bed erosion around the float open caissons are complex due to the three-dimensional character of flow-sediment movement and the randomness of movable river boundary. So far, few mathematical models are applied to describe it directly, and the experimental methods are adopted mainly[1].

Basing on the principle of sediment transportation and river bed deformation, analysing the relation on float open caisson settlement and river bed erosion, the main causes of river bed deformation lie in: (1) the float open caissons contract the flow and increase the unit discharge when settling, (2) the resistant function to the flow that results in the change of flow structure, (3) non-equilibrium transport of suspended load that results in the change of sediment concentration. The sediment concentration in flow and the scouring and silting of river bed will reach equilibrium, the river bed gradually forms erosion-resisting protective layer. We define this equilibrium state as the limit state of river bed erosion. For the limit state of river bed erosion, if the main cause of river bed deformation is the increment of unit discharge causing the increment of local velocity on river bed, after river bed is eroded, the velocity of flow is below the component velocity for sedimentation on the surface of river bed,

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river bed armors itself and forms erosion-resisting protective layer, this equilibrium state is the static balance state. The described model is clear water flow mathematical model, and suspend sediment is not taken into consideration, the main control factor is the comparison of the velocities of solid-liquid phases on the surface of river bed. Furthermore, if the amount of incoming sediment from the upper reaches is not consistent with the carrying capacity of the reach under examination, then deformation of the river channel occurs to replenish and adjust sediment concentration in flow of the channel. Adjusted sediment concentration will be harmonious with incoming sediment, and the new equilibrium state is defined as movable balance state[2].

Whether the clear water flow or the sediment-laden flow mathematical model is applied is based on the change of boundary conditions and the water-sediment factor from the upper reaches. Before the float open caisson get to the bottom of the river, it settles slowly and intermittently, the main causes of river bed deformation are the increment of unit discharge and the change of flow structure around the float open caisson, the clear water flow mathematical model can be applied in calculation, and the flow upstream can be considered as flow with no sediment or the flow with wash load. After the float open caissons enter the river bed, the constructive period is long relative to the time of entering into the water, the main causes of river bed deformation are change of flow and sediment transportation upstream, and the sediment-laden flow mathematical model must be applied[3].

2 BASIC GOVERNING EQUATIONS OF WATER-SEDIMENT VARIABLE-DENSITY TURBULENCE

River flow is variable-density turbulent in which the density of water-sediment system changes with time and space, and there is synchronism of change of sediment concentration and sediment supplement from river bed. In the category of continuum mechanics, it abides by conservation of mass, momentum and energy, here the change of temperature field is not considered, so the basic governing equations are the mass conservation equation and momentum conservation equation[4]

$$\frac{\partial \tilde{\rho}_m}{\partial t} + \frac{\partial}{\partial x_j}(\tilde{\rho}_m \tilde{u}_j + \rho'_m u'_j) = 0$$

$$\frac{\partial}{\partial t}(\tilde{\rho}_m \tilde{u}_i) + \frac{\partial}{\partial x_j}(\tilde{\rho}_m \tilde{u}_i \tilde{u}_j) = \tilde{\rho}_m \tilde{f}_i - \frac{\partial \tilde{P}}{\partial x_i} + \frac{\partial}{\partial x_j}(\tilde{\rho}_m u'_i u'_j - \tilde{u}_j \rho'_m u'_i - \tilde{u}_i \rho'_m u'_j)$$

$$\frac{\partial}{\partial t}(\tilde{\rho}_m k) + \frac{\partial}{\partial x_j}(\tilde{\rho}_m \tilde{u}_j k) = \frac{\partial}{\partial x_j}\left(\frac{\mu_e}{\sigma_k} \frac{\partial k}{\partial x_j}\right) + \tilde{\rho}_m (G_k - \varepsilon)$$

$$\frac{\partial}{\partial t}(\tilde{\rho}_m \varepsilon) + \frac{\partial}{\partial x_j}(\tilde{\rho}_m \tilde{u}_j \varepsilon) = \frac{\partial}{\partial x_j}\left(\frac{\mu_e}{\sigma_\varepsilon} \frac{\partial \varepsilon}{\partial x_j}\right) + \tilde{\rho}_m \varepsilon (c_1 G_k - c_2 \varepsilon)$$

where, $\tilde{\rho}_m$ is time average density of variable-density flow, $\rho'_m$ is fluctuating value of time average density, $\tilde{u}_i$ is time average velocity in $x$, $y$ and $z$ direction, $\tilde{f}_i$ is mass force, $\tilde{P}$ is time average pressure, $k$ and $\varepsilon$ are turbulent energy and ratio of turbulent dissipation, $\mu_e$ is dynamic viscosity coefficient of variable-density, $G_k$ is generation item of turbulent energy, $\sigma_k$, $\sigma_\varepsilon$, $c_1$ and $c_2$ are model coefficients.

Transforming the sediment-laden flow density in the above equations into the solid division density of two-phase flow, i.e. sediment concentration $s$, applying tensor analysis[5] of fluid, the equations in the general curve coordinates can be written (omitting the "-"
above the time average values).

Flow continuity equation
\[
\frac{\partial u_i}{\partial x_i} = \frac{1}{J} b_i^j \frac{\partial u_j}{\partial \xi^j} = 0
\]  
(5)

and other control equations
\[
a_{\phi} R_{\text{eff}} \frac{\partial \phi}{\partial t} - g^{ij} \frac{\partial \phi}{\partial \xi^i} \frac{\partial \phi}{\partial \xi^j} = 2A^i \frac{\partial \phi}{\partial \xi^i} + s_{\phi}
\]  
(6)

where \( \phi = (u, v, w, s, k, \varepsilon)^T, a_{\phi}, A^i \) and source term \( s_{\phi} \) are
\[
a_u = a_v = a_w = 1 \quad a_s = 1/(U_0 \beta) \quad a_k = \sigma_k \quad a_{\varepsilon} = \sigma_{\varepsilon}
\]  
(6a)
\[
2A^i = \frac{1}{J} R_{\text{eff}} \left[ a_{u} b_i^j u_j - \frac{1}{J} b_i^j b_j^k \frac{\partial \gamma_k}{\partial \xi^i} - \frac{\partial}{\partial \xi^i} (J g^{ij}) \right]
\]  
(6b)
\[
s_{ui} = \frac{1}{J} R_{\text{eff}} \left[ b_i^k \frac{\partial P}{\partial \xi^k} + \frac{2}{3} b_i^j b_j^k \frac{\partial k}{\partial \xi^i} - \frac{1}{J} b_i^j b_j^k \frac{\partial u_j}{\partial \xi^k} \frac{\partial \gamma_l}{\partial \xi^i} - u_i b_i^j (\beta - b_j^j u_j) b_j^l \frac{\partial s}{\partial \xi^l} - b_i^j u_j \beta b_i^k \frac{\partial s}{\partial \xi^k} \right]
\]  
(6c)
\[
s_s = \frac{S_{\text{eff}}}{\beta} \frac{1}{J} b_i^j \frac{\partial u_j}{\partial \xi^i}
\]  
(6d)
\[
s_k = -\sigma_k R_{\text{eff}} \left( G_k - \varepsilon - b_j^j u_j k b_i^i \frac{\partial s}{\partial \xi^i} \right)
\]  
(6e)
\[
s_s = -\sigma_s R_{\text{eff}} \left[ \varepsilon \left( c_1 G_k - c_2 \varepsilon \right) - b_j^j u_j \varepsilon b_i^i \frac{\partial s}{\partial \xi^i} \right]
\]  
(6f)

\( R_{\text{eff}} \) and \( S_{\text{eff}} \) are dimensionless Reynolds and sediment Reynolds number, \( \nu_t \) is turbulent viscosity coefficient, \( \xi^i \) is coordinate component of the general curve coordinate system, \( b_i^j \) is matrix of ortho-transformation algebraic complement, \( g^{ij} \) is the inverse matrix of measurement tensor, \( J \) is the absolute value of ortho-transformation coefficient, \( U_0 \) is dimensionless character velocity, \( \beta \) is model coefficient of sediment equation.

3 BOUNDARY CONDITIONS OF MATHEMATICAL MODEL FOR CLEAR WATER FLOW AND SEDIMENT-LADEN FLOW

On the basis of the characters of clear water flow and sediment-laden flow models, it is recognized that the differences of two kinds of models lie in the constitution of boundary conditions for solid-phase around river bed. But the boundary conditions are the same for the liquid phase. On upstream boundary, Dirichlet boundary condition is applied to all variables except for pressure, for which Neumann boundary condition is derived from momentum equation. On downstream boundary, Neumann boundary condition is applied to all variables. Symmetric boundary condition and the dynamic boundary condition are applied on symmetric boundary and free surface boundary respectively. On solid boundary, three-dimensional wall function of flow is a simple and effective method.

For the clear water flow mathematical model, the river bed boundary conditions are component velocity, in which the coarse sediment shadow and block the fine sediment to form the protective-erosion layer. For the sediment-laden flow mathematical model, sediment concentration in boundary layer of river bed involves the exchange with bed load, and deformation of the bed involves change of the unsaturated sediment capability.
Sediment concentration in boundary layer of the river bed can be represented by average concentration in the concentration boundary layer, which is referred to the very small zone adjacent to the river bed with relatively high concentration.

\[ S_a = \frac{\rho g (h - y_b) \sin \theta - \tau_b}{(\rho_s - \rho)(a - y_b)g(\cos \theta \tan \phi - \sin \theta)} \quad (7) \]

where, \( \rho \) and \( \rho_s \) are liquid and solid density respectively, \( g \) is gravitation acceleration, \( h \) is water depth, \( y_b = 0.8d \), \( d \) is diameter of bed load, \( a \) is the thickness of concentration boundary layer, \( \theta \) is angle of intersection grid, \( \phi \) is equal to the angle of friction of bed load, \( \tau_b \) is resistant of river bed surface.

Introducing the concept of recovery coefficient that is widely applied in China, river bed deformation caused by unsaturated sediment transport is expressed as

\[ \rho' \frac{\partial Z_{sk}}{\partial t} = \alpha \omega_k (S_k - S_{sk}) \quad (k = 1, 2, \cdots, N_s) \quad (8) \]

where \( \rho' \) is dry bulk density of bed load, \( Z_{sk} \) denotes the river bed deformation caused by the \( k \)-th fraction sediment, \( \omega_k \) is the settling velocity of the \( k \)-th fraction sediment, if the flow is with higher concentration of sediment, it can be readjusted according to Ref.[7], \( S_k \) and \( S_{sk} \) are sediment concentration and sediment-carrying capacity of the \( k \)-th grain size fraction respectively, \( \alpha \) is recovery coefficient.

For determining the sediment-carrying capacity of different grain size, a number of studies have been carried out, using Zhang Ruijing’s formula, we have

\[ S_{sk} = P_{bk} K \left( \frac{U^3}{gR \omega_k} \right)^m \quad (9) \]

and

\[ P_k = \left( \frac{P_{bk}}{\omega_k^m} \right) / \sum_{k=1}^{N_s} \frac{P_{bk}}{\omega_k^m} \quad (10) \]

where \( R \) is hydraulic radius, \( U = \sqrt{u^2 + v^2 + w^2} \), \( K(U^2/(gR\omega_k))^m \) is possible carrying of uniform sediment with an equivalent settling velocity \( \omega_k \) of the \( k \)-th grain size fraction, \( P_{bi} \) is the size distribution of bed material, that is the percentage of the \( k \)-th fraction of bed material, \( P_i \) is the size distribution of sediment transported at the sediment-carrying capacity, \( K \) and \( m \) are model coefficients.

4 CALCULATION EXAMPLE

All the above research have been applied to the erosion calculation of the float open caisson when it enters water and river bed. Before entering river bed, the parts of the float open caissons are connected gradually. In the process of connection, there is a stable period in the float open caisson, flow and sediment, so that the mathematical model of steady flow can be applied in that period. After entering the river bed, taking into account the restriction of calculation time and computer capability, the upstream flow and sediment can be assumed to be a gradual change process, we can consider the discharge of flow and sediment concentration as step process, in which every step is steady process. The model equations include flow continuity equation, flow momentum equations, \( k \) and \( \epsilon \) equations, component velocity equations of non-uniform sediment. After the float open caisson entering river
bed, we can use mathematical model of sediment-laden flow which includes continuity equations of flow and solid-phase (sediment), momentum equation of sediment-laden flow, k and ε equation as well as bed deformation equation. Total numerical solution procedure of the settling process of float open caisson construction period includes:

1. Setting up of the boundary fitted coordinate system for the total solution domain of float open caisson, calculate the transformation coefficient.

2. Solution of the simultaneous equations of continuity equation, momentum equation, k and ε equation (including solid-phase sediment continuity equations in sediment-laden flow), calculation of velocity field, pressure field and free surface of flow.

3. Calculation of the erosion and armor of grid at river bed for clear water flow according to the results of velocity field and pressure field, verifying the river bed boundary and adjusting the gradation of bed load. Solution of bed deformation equation for sediment-laden flow, obtaining of the results of river form and adjustment of the gradation of bed load.

4. Rebuilding of the boundary fitted coordinate systems according to the new surface of flow and boundary of river bed, repeating procedure (2) and (3). In flow mathematical model, when the error in component velocity of sediment in grid is below the allowable range, prints the results. In sediment model, when the sediment concentration in boundary layer of river bed is below the allowable range, print the results.

In this paper, we calculate the flow, sediment transport and river bed deformation, the results are shown in Figs.1~5. Analyzing the calculation results, we can find that

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**Fig.1** The z-y plane velocity field at 7.5 m

**Fig.2** The z-z plane velocity field at 7.5 m

**Fig.3** The erosion topography of calculation at 7.5 m
Fig. 4  The topography corresponding different process of water-sediment movement when entering river bed

Fig. 5  Comparison of the calculation and measurement results at 7.5m (in x-y and z-x plane)

the flow structure will experience a series of changes with the different height of the float open caisson settlement. Velocity field of cross section changes much more evidently than surface velocity. The deeper the float open caisson descends, the more becomes the flow resistance produced by the float open caisson, and the wider becomes the range that flow open caisson effects. The bottom flow is disturbed and the river bed is eroded. River bed forms the protective layer of erosion. The shape of erosion cavity is similar to the U-shape.
When the caisson enters the bottom of the river bed, the erosion flow is similar to the erosion of the bridge pier which will erode with flood and silt with recession. Figure 5 compares the calculation and measurement results at the 7.5 m of the float open caisson settlement. The maximum error is less than 1.0 m which shows that the above mathematical model possesses good accuracy.

5 CONCLUSION

It is of theoretical and practical importance to apply the mathematical models of clear water flow and sediment-laden flow for variable-density two-phase flow to the bridge and hydraulic engineering with local complex boundaries. In comparison with the mathematical models of sediment-laden flow, the clear water flow mathematical models have developed more thoroughly, and provided more accurate results. The sediment-laden mathematical model has more complex mechanism of stress relation for two-phase flow, exchange of river bed boundary and sediment capability. The measurement can not satisfy some flow and sediment transport relations. With the deeper knowledge of the mechanism and accumulation of experiences in numerical calculation, the mathematical model of two-phase flow will be perfected gradually.

REFERENCES